Day 19

Combining Noisy Measurements

Combining Noisy Measurements

- suppose that you take a measurement x_1 of some real-valued quantity (distance, velocity, etc.)
- your friend takes a second measurement x_2 of the same quantity
- after comparing the measurements you find that

$$x_1 \neq x_2$$

- what is the best estimate of the true value μ?

Combining Noisy Measurements

- suppose that an appropriate noise model for the measurements is

$$x_1 = x + \varepsilon_{\sigma^2}$$

$$x_2 = x + \varepsilon_{\sigma^2}$$

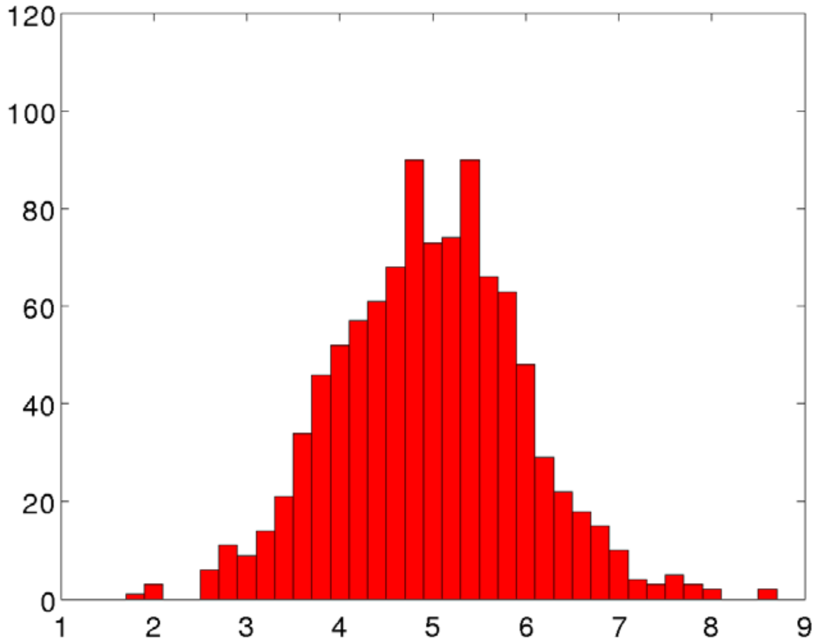
  where ε_{σ^2} is zero-mean Gaussian noise with variance σ^2
- because two different people are performing the measurements it might be reasonable to assume that x_1 and x_2 are independent

Combining Noisy Measurements

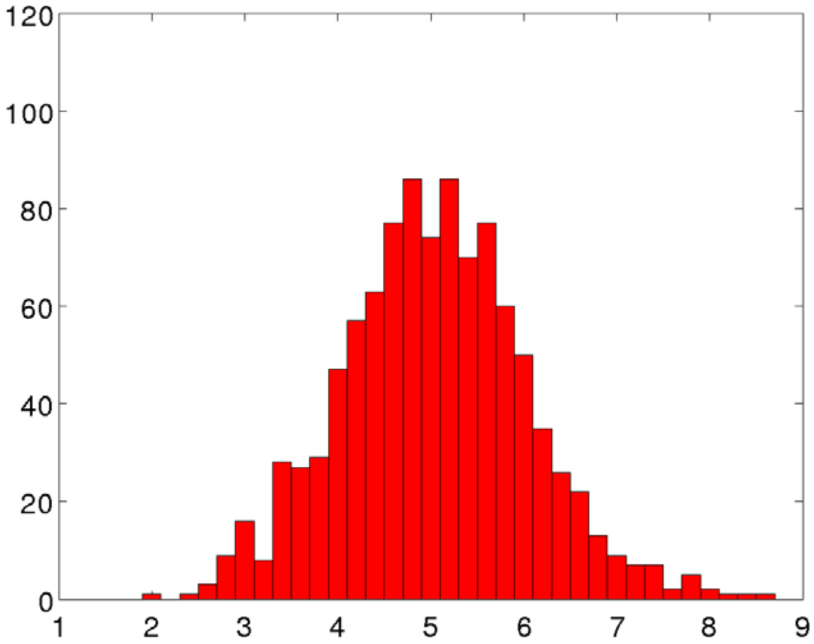
```
x = 5;
x1 = x + randn(1, 1000);   % noise variance = 1
x2 = x + randn(1, 1000);   % noise variance = 1
mu2 = (x1 + x2) / 2;

bins = 1:0.2:9;
hist(x1, bins);
hist(x2, bins);
hist(mu2, bins);
```

Combining Noisy Measurements

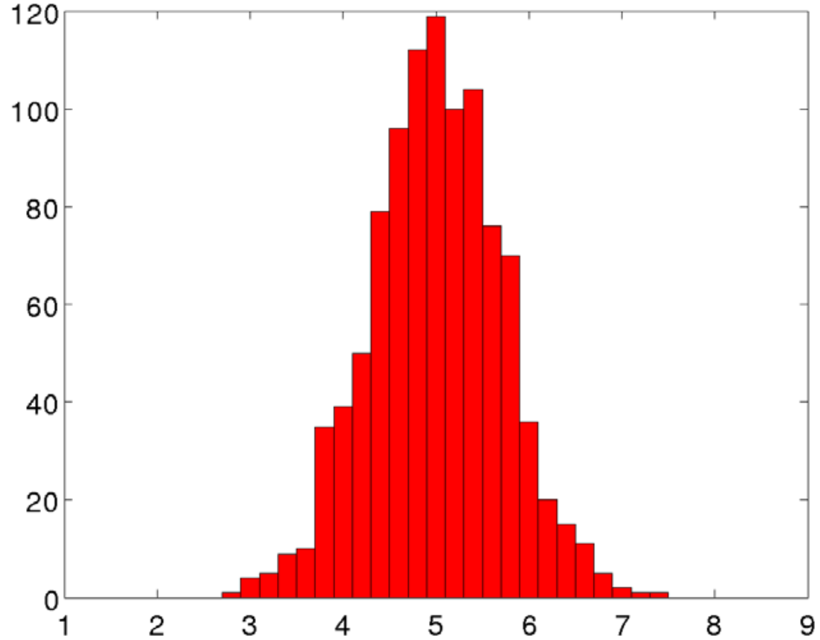


var(x1) = 0.9979



var(x2) = 0.9972

Combining Noisy Measurements



var(x1) = 0.4942

Combining Noisy Measurements

- suppose the precision of your measurements is much worse than that of your friend
- consider the measurement noise model

$$x_1 = x + 3\varepsilon_{\sigma^2}$$

$$x_2 = x + \varepsilon_{\sigma^2}$$

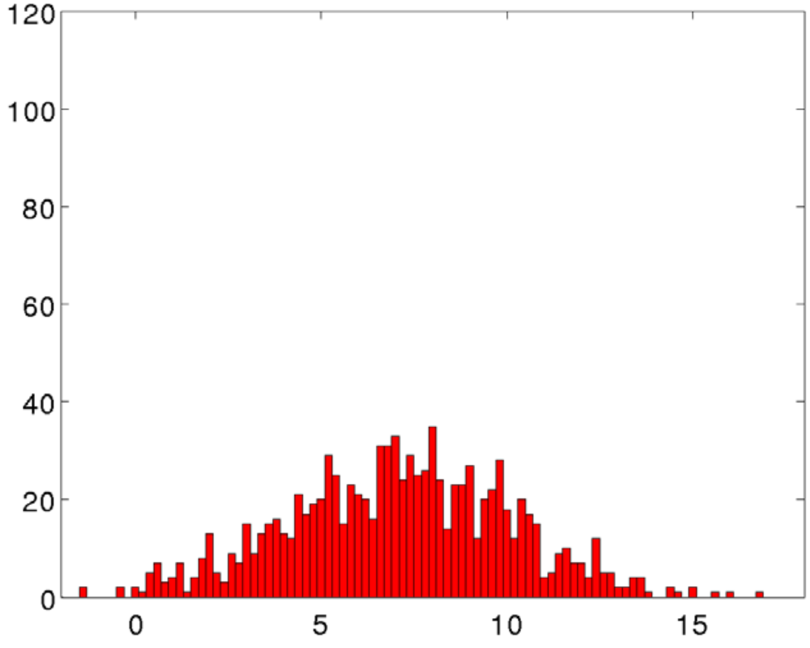
where ε_{σ^2} is zero-mean Gaussian noise with variance σ^2

Combining Noisy Measurements

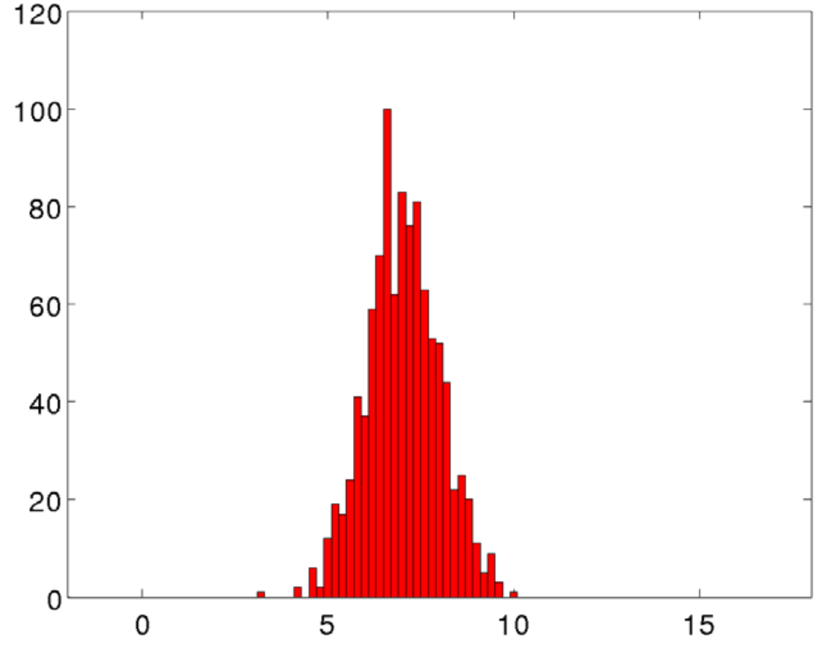
```
x = 7;
x1 = x + 3 * randn(1, 1000);   % noise variance = 3*3 = 9
x2 = x + randn(1, 1000);       % noise variance = 1
mu2 = (x1 + x2) / 2;

bins = -2:0.2:18;
hist(x1, bins);
hist(x2, bins);
hist(mu2, bins);
```

Combining Noisy Measurements

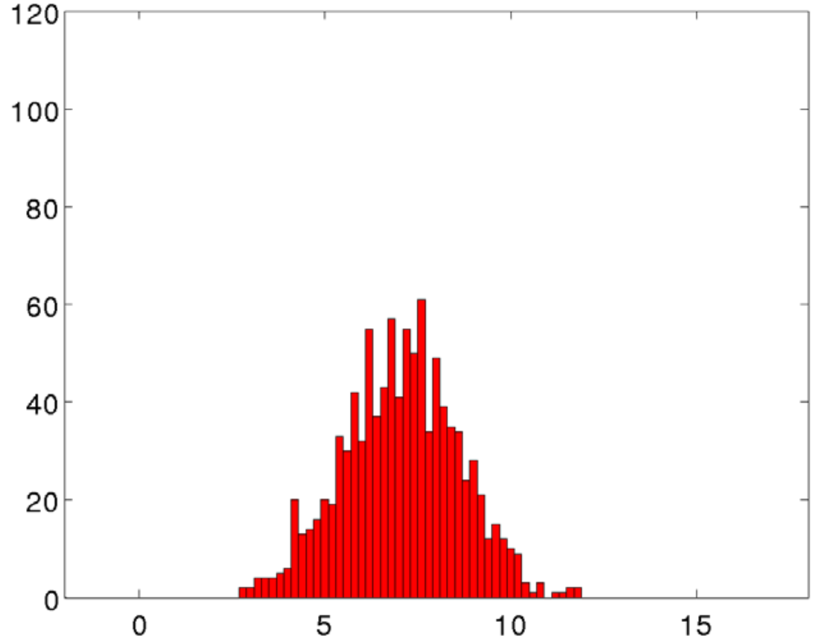


var(x1) = 8.9166



var(x2) = 0.9530

Combining Noisy Measurements



var(mu2) = 2.4317

Combining Noisy Measurements

- is the average the optimal estimate of the combined measurements?

Combining Noisy Measurements

- instead of ordinary averaging, consider a weighted average

$$\mu = \omega_1 x_1 + \omega_2 x_2$$

where $\omega_1 + \omega_2 = 1$

- the variance of a random variable is defined as

$$\mathrm{var}(X) = \mathrm{E}[X - \mathrm{E}[X]]^2$$

where $\mathrm{E}[X]$ is the expected value of X

Expected Value

- informally, the expected value of a random variable X is the long-run average observed value of X
- formally defined as

$$\mathrm{E}\left[X\right] = \int_{-\infty}^{\infty} x \; f(x) \, dx$$

- properties

$$\mathrm{E}\left[c\right] = c$$
$$\mathrm{E}\left[\mathrm{E}\left[X\right]\right] = \mathrm{E}\left[X\right]$$
$$\mathrm{E}\left[X + c\right] = \mathrm{E}\left[X\right] + c$$
$$\mathrm{E}\left[X + Y\right] = \mathrm{E}\left[X\right] + \mathrm{E}\left[Y\right]$$
$$\mathrm{E}\left[cX\right] = c\mathrm{E}\left[X\right]$$

Variance of Weighted Average

$$
\begin{aligned}
\mathrm{var}(\mu) &= \mathrm{E}[(\mu - \mathrm{E}[\mu])^2] \\
&= \mathrm{E}[(\omega_1 x_1 + \omega_2 x_2 - \mathrm{E}[\omega_1 x_1 + \omega_2 x_2])^2] \\
&= \mathrm{E}[(\omega_1 x_1 + \omega_2 x_2 - \omega_1 \mathrm{E}[x_1] - \omega_2 \mathrm{E}[x_2])^2] \\
&= \mathrm{E}[((\omega_1 (x_1 - \mathrm{E}[x_1]) + \omega_2 (x_2 - \mathrm{E}[x_2]))^2] \\
&= \mathrm{E}[\omega_1^2 (x_1 - \mathrm{E}[x_1])^2 + \omega_2^2 (x_2 - \mathrm{E}[x_2])^2 + \\
&\qquad 2\omega_1\omega_2 (x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])] \\
&= \omega_1^2 \mathrm{E}[(x_1 - \mathrm{E}[x_1])^2] + \omega_2^2 \mathrm{E}[(x_2 - \mathrm{E}[x_2])^2] + \\
&\qquad 2\omega_1\omega_2 \mathrm{E}[(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])] \\
&= \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 + 2\omega_1\omega_2 \mathrm{E}[(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])]
\end{aligned}
$$

Variance of Weighted Average

- because x_1 and x_2 are independent

$$(x_1 - \mathrm{E}[x_1]) \quad \text{and} \quad (x_2 - \mathrm{E}[x_2])$$

are also independent; thus

$$\mathrm{E}[(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])] = 0$$

- finally

$$\mathrm{var}(\mu) = \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2$$

Variance of Weighted Average

- because x_1 and x_2 are independent

$$(x_1 - \mathrm{E}[x_1]) \quad \text{and} \quad (x_2 - \mathrm{E}[x_2])$$

are also independent; thus

$$\mathrm{E}[(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])] = 0$$

- finally

$$\begin{aligned}\mathrm{var}(\mu) &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 \\ &= (1-\omega^2)\sigma_1^2 + \omega^2\sigma_2^2 \quad \text{where} \quad \omega_2 = \omega, \quad \omega_1 = 1-\omega\end{aligned}$$

Variance of Weighted Average

- one way to choose the weighting values is to choose the weights such that the variance is minimized

$$\frac{d}{d\omega}\mathrm{var}(\mu) = 0 = -2(1-\omega)\sigma_1^2 + 2\omega\sigma_2^2$$

$$\Rightarrow \omega = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

Minimum Variance Estimate

- thus, the minimum variance estimate is

$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} x_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} x_2$$

$$\mathrm{var}(\mu) = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

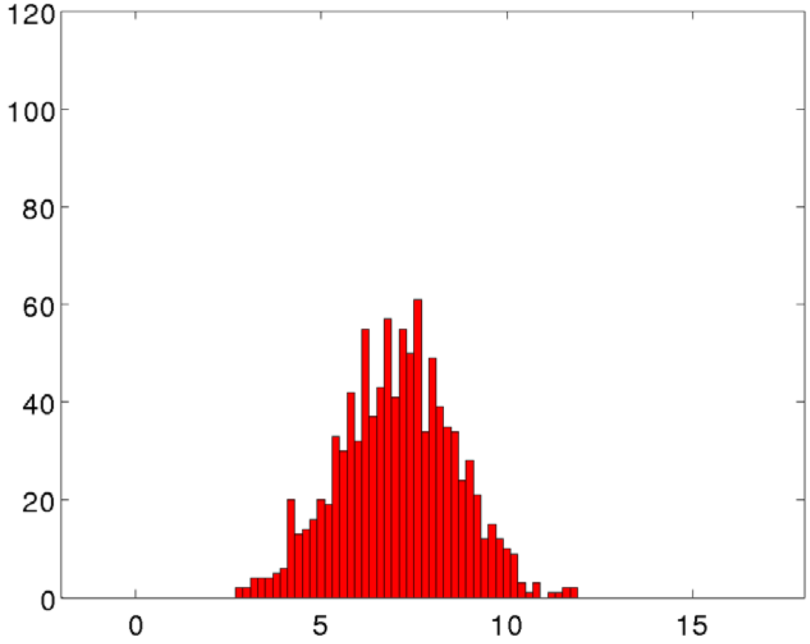
Combining Noisy Measurements

```
x = 7;
x1 = x + 3 * randn(1, 1000);   % noise variance = 3*3 = 9
x2 = x + randn(1, 1000);       % noise variance = 1
w = 9 / (9 + 1);
mu2 = (1 – w) * x1 + w * x2;

bins = -2:0.2:18;
hist(x1, bins);
hist(x2, bins);
hist(mu2, bins);
```

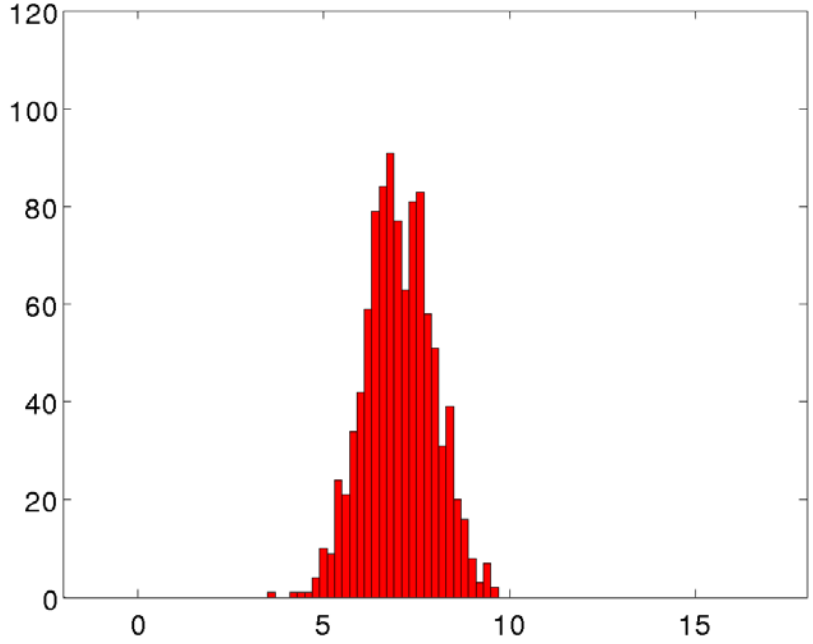
Minimum Variance Estimate

mu2=0.5*x1 + 0.5*x2



var(mu2) = 2.4317

mu2=0.1*x1 + 0.9*x2



var(mu2) = 0.8925